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LETTER TO THE EDITOR

Eden model on the Manhattan lattice

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Abstract. We study the Eden growth model on the square Manhattan lattice (SML). For this purpose it has been necessary to extend the standard Eden model in such a way that each act of growing is conceived as a directional process. By applying the postition space renormalisation group technique, we find that the Eden processes on the SML and on the ordinary square lattice are in different universality classes. This finding and corresponding fractal dimensions are compared with results obtained for similar models of irreversible kinetic processes.

There has been vigorous interest in the physical description of the structure of aggregates, or clusters, formed by irreversible kinetic processes. The latter include processes, such as gelation and coagulation, which are relevant to various physical, chemical, biological and technological phenomena. It has been concluded (see, for example, Stanley 1983) that objects (aggregates) formed by these processes may have scale invariant structures and can be described as fractals (Mandelbrot 1982). It means that the number of units (particles) N of an aggregate and its mean square radius $\langle R_N^2 \rangle$ are asymptotically (for large N) related by

$$N^2 \sim \langle R_N^2 \rangle^D \tag{1}$$

where D is the Hausdorff or fractal dimension of the aggregate. If D is less than the spatial dimension d, the aggregates are ramified, whereas for $D = d$ they are termed compact. For a growing process D is its statistical quality. That is to say, in the corresponding ensemble of aggregates, a needle-like cluster and a compact ball may appear, but the shape of the largest number of clusters should be in accord with D that is characteristic of the particular growing process. Thus D has been used to classify the growth models of randomly formed aggregates into appropriate universality classes (see, for example, Gould *et al* 1983, Green 1984).

In this letter we study the Eden growth model (Eden 1961), where cluster growth is effected in a simple way, by adding particles at random, with uniform probability, on the boundary of the cluster. If the particles are to occupy sites of a lattice, clusters formed in the Eden process look like growing animals (Nakanishi and Family 1984). The numerical simulation (for $d = 2, 3$) of the aggregation process (Peters *et al* 1979) has shown that the fractal dimension of the Eden aggregates is equal to the spatial (or embedding) dimension d. This implies that the aggregates have, on average, a very

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compact structure. The result is plausible, since all interior vacant patches in a cluster will eventually be smeared, as their perimeter sites will sooner or later be chosen for occupation. However, although one may be tempted to accept the numerical finding for arbitrary d, Parisi and Zhang (1984) have shown that in the limit when d tends to infinity, $\langle R_N^2 \rangle$ shows asymptotic behaviour which is not consistent with the compact-clusters picture. Their result seems to agree with the results of Vannimenus *et al* (1984), who have studied the Eden model on the Cayley tree. On the other hand, the position space renormalisation group (PSRG) approach of Gould *et al* (1983) gave quite good results for the diffusion-limited aggregation (DLA) model proposed by Witten and Sander (1981), but for the Eden model it failed to provide D close enough to d (when $d = 2, 3$). Taken together these facts suggest that the Eden model is not as simple as it appeared to be, and hence it deserves further investigations.

The particular question that we seek to answer concerns the universality classes of the Eden model on the ordinary square lattice and on the square Manhattan lattice. We shall use the PSRG method. This method has been applied to the problem of the universality classes of the DLA growth models and their equilibrium counterparts (Gould *et al* 1983, Green 1984). However, the problem we are concerned with is in fact the search of the dynamic universality classes, since we are going to investigate the single growth process on two different substrata. To this end, we shall use the PSRG technique introduced by Prentis (1984), and elaborated by Malakis (1984), for the self-avoiding-walks (SAW) problem. Within the accepted framework we shall reach the conclusion that the Eden aggregates on the ordinary square lattice and on the square Manhattan lattice are in different universality classes. This result is proposed for discussion.

The square Manhattan lattice (SML) is a two-dimensional oriented square lattice (see figure 1), whose network of bonds resembles the traffic scheme of Manhattan downtown. On a large scale, the SML is an isotropic system. However, in order to place the Eden growth process on the SML, we have to extend the Eden model in such a way that each act of growth should be directed along a single bond that connects the new particle with a particle contained in the aggregate perimeter. For example, the four-particle aggregate (see figure 2(a)) can be uniquely grown out of the three-particle cluster 1-2-3, in the case of the standard Eden model. On the other hand, in the case of the extended Eden model, the four-particle cluster can be formed out of the 1-2-3 cluster, either by adding the fourth particle along the bond that links it with the second particle, or along the bond that links it with the third particle (see figure 2(b)). Hence, in the ensemble of clusters pertinent to the extended Eden model, the four-particle cluster should be counted twice. Before continuing with the technical

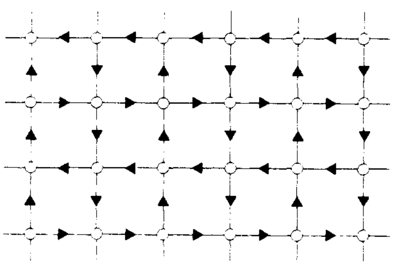

Figure 1. The square Manhattan lattice.

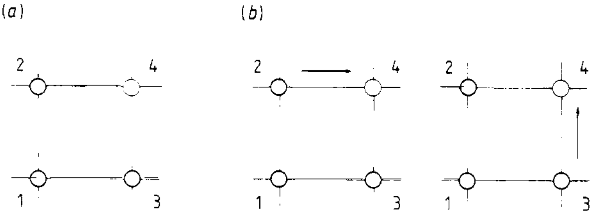

Figure 2. (*a*) According to the standard Eden model the fourth particle can join the preceding three in a single way. (*b*) The extended Eden model allows of the two ways of aggregating the fourth particle.

details, we point out that the extended Eden model, in comparison with the standard model should be more relevant to those real processes in which the probability of appearance of a new element in the aggregate is larger at those places where there exist more contacts with already aggregated elements. Such is the process of the spreading of a fire in a forest, or of diseases in a orchard.

If the two Eden models (standard and extended) are placed on the ordinary square lattice, one can expect to see growing of aggregates characterised by the same fractal dimension $D=2$. Indeed, accepting the numerical finding (Peters *et al* 1979) that the aggregates of the standard Eden model should be compact ($D=d$), one can provide a qualitative argument that the aggregates of the extended Eden model cannot be less compact. The argument stems from the fact that unoccupied sites close to the border of an interior vacant patch in the aggregate have, on average, more already aggregated neighbours than sites close to the circumference of the cluster; therefore the vacant patches of the aggregates grown in the extended Eden process should more rapidly vanish than those which appear within the aggregates grown in the standard Eden process. We support this argument by a one-parameter PSRG analysis of the type suggested by Gould *et al* (1983). These authors place a particle at an initial seed site and allow the cluster to grow only eastwards and northwards, associating a weight, or fugacity, K with each occupied site in the cluster. Next, a rescaling of the lattice is performed via the cell-to-site mapping, which means that the lattice is divided into cells of linear dimension b and the cells are rescaled to single site, with the corresponding fugacity K'. Finally, the renormalisation group (RG) transformation is introduced

$$K' = R(K) \tag{2}$$

where $R(K)$ includes all the spanning configurations that can be grown from the initial seed site. The fractal dimension D is given by

$$D = \ln \lambda_K / \ln b \tag{3}$$

where λ_K is the eigenvalue, $\lambda_K = (\partial K'/\partial K)_{K=K_c}$, of the transformation, with K_c being its critical fixed point (Stanley *et al* 1982). Thereby, in the case of the extended Eden model, we find (for $b=2$) the following RG equation

$$K' = 4K^3 + 8K^4 \tag{4}$$

which gives $K_c = 0.377$ and $D = 1.778$. These values should be compared with the results $K_c = 0.420$ and $D = 1.721$, obtained for the standard Eden model (Gould *et al* 1983), whereupon one can see that the clusters of the extended Eden model cannot be less compact than the clusters of the standard Eden model. To support this conclusion, and for the sake of a later analysis, we give here the RG equation for the extended Eden model

$$K' = 22K^5 + 318K^6 + 2128K^7 + 7748K^8 + 17\,528K^9 \qquad (5)$$

in the case $b = 3$. The latter equation gives $K_c = 0.231$ and $D = 1.7991$, which should be compared with the values $K_c = 0.279$ and $D = 1.729$, found for the standard Eden model (Gould *et al* 1983).

Now we place the extended Eden model on the SML; henceforth we refer to the model, whenever it does not cause any ambiguity, just as the Eden model. We first perform the one-parameter PSRG analysis of the type described in the preceding paragraph, allowing that each act of growth occurs only in the direction of the lattice bond that connects two neighbouring sites. The problem to be solved concerns the four types of cells that appear on the SML when $b = 3$. In the case of the SAW on the same lattice (Prentis 1984, Malakis 1984) it is reasonable to study the ensemble of SAW which are upward oriented, and thus only two types of cells are relevant in the corresponding cell-to-bond mapping. Furthermore, if one uses the 'equal averaging' rule instead of the 'corner rule' (see, for example, Redner and Reynolds 1981), the resulting RG transformations appear to be the same for both cells. However, in the case of the Eden model, with the accompanied cell-to-site mapping, there is no possibility of getting a unique transformation for the four different cells. Therefore, we resort to the arithmetic mean

$$K' = \tfrac{1}{4}(16K^5 + 66K^6 + 162K^7 + 292K^8 + 376K^9) \qquad (6)$$

which gives $K_c = 0.434$ and $D = 1.7596$. The latter value differs from $D = 1.7991$ found for the extended Eden model on the ordinary square lattice. Yet, because of the approximate nature of the PSRG technique, the observed difference does not automatically imply that the Eden aggregates on the SML and on the ordinary square lattice are in different universality classes.

To resolve the problem of the universality classes of the Eden aggregates we next apply the two-parameter PSRG technique introduced by Prentis (1984). Following Prentis (1984) and Malakis (1984), we allow that an act (step) of growth on the SML may violate the underlying bond direction with a probability $(1-p)$. Accordingly, a step of growth obeying the bond direction may be performed with the probability p. Within the PSRG approach, we regard these probabilities as appropriate weights of acts of growing a cluster, while the parameter K (fugacity) remains the weight of each particle in the cluster. Hence, for example, the weight of a l-site cluster is $K^l p^s (1-p)^r$ if the cluster has been grown through s steps that obey the underlying orientations of the bonds and through r steps that violate the underlying orientations, so that $s + r = l - 1$. As regards the introduced cluster weight there are two immediate comments. First, the weighting of clusters (on the SML) is unambiguous due to the accepted extension of the Eden model. Second, the above-mentioned weighting differs from the weighting used in dealing with the SAW problem (Prentis 1984), where each self-avoiding-walk step is weighted either by the product pK (if the step obeys the underlying orientation of the bond) or by $(1-p)K$ (in the opposite case). This difference springs from the twofold nature of the Eden model, as it brings on both a site problem, in which essential objects are clusters of occupied sites, and the problem of a kinetic process, where the growth of clusters occurs via bonds of the underlying lattice.

Changing the parameter p from 1 to 0.5 (or from 0 to 0.5) transforms the underlying lattice from the SML (from the anti-Manhattan lattice) to the ordinary square lattice. The studied problem should be symmetrical with respect to the choice of the probability parameter (p or $q=1-p$) and so should be the corresponding RG transformations. To introduce the renormalised probability p' and the renormalised fugacity K' we look at two rescaled sites which are linked with a rescaled bond, and assign to the bond an orientation that is determined by a majority rule. Specifically, in the case displayed in figure 3, we assume that the growth from the rescaled site A to the rescaled site C

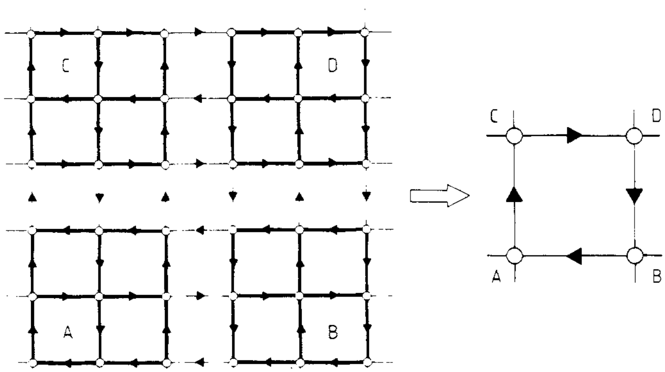

Figure 3. Rescaling of the SML by forming cells out of groups of sites and by renormalising the bond orientations according to majority rule.

obeys the underlying direction of the rescaled bond (and occurs with the probability p'), whereas the growth from A to B violates (with the probability $1-p'$) the relevant bond direction. Thus, we form the corresponding RG equations

$$p'K'^2 = pF_A(K, p, 1-p)F_C(K, p, 1-p) \tag{7}$$

$$(1-p')K'^2 = (1-p)F_A(K, p, 1-p)F_B(K, p, 1-p) \tag{8}$$

where F_A, F_B and F_C are the total weights of all spanning clusters that can be grown on the cells A, B, and C respectively. The prefactor p and similarly $1-p$ on the right-hand side of equations (7) and (8) respectively correspond to the possibility of growing a cluster on cell C or cell B respectively from a cluster on the cell A. By enumerating all possible clusters we have found the following explicit expressions;

$$\begin{aligned} F_A(K, x, y) = {} & K^5(4x^4+2x^3y+10x^2y^2+2xy^3+4y^4) \\ & + K^6(16x^5+57x^4y+86x^3y^2+57xy^4+16y^5) \\ & + K^7(25x^6+209x^5y+510x^4y^2+640x^3y^3+510x^2y^4+209xy^5+25y^6) \\ & + K^8(29x^7+354x^6y+1326x^5y^2+2165x^4y^3+2165x^3y^4+1326x^2y^5 \\ & +354xy^6+29y^7) \\ & + K^9(274x^7y+1824x^6y^2+4072x^5y^3+5188x^4y^4 \\ & +4072x^3y^5+1824x^2y^6+274xy^7), \end{aligned} \tag{9}$$

$$F_C(K, x, y) = K^5(8x^4 + 10x^2y^2 + 4xy^3) + K^6(34x^5 + 60x^4y + 132x^3y^2 + 80x^2y^3 + 12xy^4) + K^7(112x^6 + 344x^5y + 694x^4y^2 + 720x^3y^3 + 238x^2y^4 + 20xy^5) + K^8(234x^7 + 838x^6y + 1902x^5y^2 + 2716x^4y^3 + 1686x^3y^4 + 352x^2y^5 + 20xy^6) + K^9(376x^8 + 1492x^7y + 3512x^6y^2 + 5408x^5y^3 + 4748x^4y^4 + 1792x^3y^5 + 200x^2y^6) \qquad (10)$$

and

$$F_B(K, x, y) = F_C(K, y, x). \qquad (11)$$

One should notice that the RG transformations (7) and (8), with F_A, F_B and F_C given by (9), (10) and (11), are invariant to the interchanging $p \leftrightarrow (1-p)$, and $p' \leftrightarrow (1-p')$, which implies that the RG transformations preserve the symmetry between the Eden growth processes on the SML and on the anti-Manhattan square lattice. Furthermore, one can also define F_D as the total weight of all possible clusters on the cell D (see figure 3), and verify that $F_D(K, x, y) = F_A(K, x, y)$. Thereby one can vindicate that the growth process, on the rescaled lattice, from D to B (northwards) and from D to C (eastwards) would bring about the same RG transformations (7) and (8). As for the three limiting cases ($p = 0, 0.5, 1$) one can check that setting $p = 0.5$ in equations (7) and (8) implies $p' = 0.5$ and retrieves equation (5) with the transformed fugacity $\frac{1}{2}K$. On the other hand, setting $p = 1$ (or $p = 0$) in equations (7) and (8) leads to $p' = 1(p' = 0)$ and

$$K'^2 = F_A(K, 1, 0)F_C(K, 1, 0) \qquad (12)$$

which shows that, in this particular case, the renormalised fugacity is the geometric mean of the cell weights. Therefore, the limiting cases should form three fixed points of the RG transformations (7) and (8). The points are determined by $p = 0.5$ and $p = 1(0)$, and by the solutions of the corresponding equations (5) and (12).

The complete renormalisation mapping governed by equations (7) and (8) is depicted in figure 4. It reveals three non-trivial fixed points. They lie on the critical manifold, which is illustrated by the full curve in figure 4. The pair of fixed points located at $(K, p) = (0.4137, 1)$ and at $(K, p) = (0.4137, 0)$ correspond to the Eden growth processes on the SML and on the anti-Manhattan square lattice respectively. The third fixed point $(\frac{1}{2}K, p) = (0.231, 0.5)$ correspond to the Eden growth process on the ordinary square lattice. Hence, the flow diagram discloses that the latter process does not belong to the universality class of the former two. This is indicated by the findings for the fractal dimensions of the corresponding aggregates ($D = 1.7991$ versus $D = 1.7302$), and above all by the fact that the point $(\frac{1}{2}K, p) = (0.231, 0.5)$ is the most unstable fixed point, with the relevant eigenvalues $\lambda_K = 7.2179$ and $\lambda_p = 8.0168$. To verify this result, we have tried several modifications of the RG transformations, stipulating that they should be compatible with the character and symmetry of the problem. For example, one could argue that the clusters on the cells A and C should be linked with the probability $(p+1)/3$, and for this reason the prefactor on the right-hand side of equation (7) should be $(p+1)/3$ rather than p. However, in all such modifications the essential character of the flow diagram (see figure 4) remain unchanged, and on these

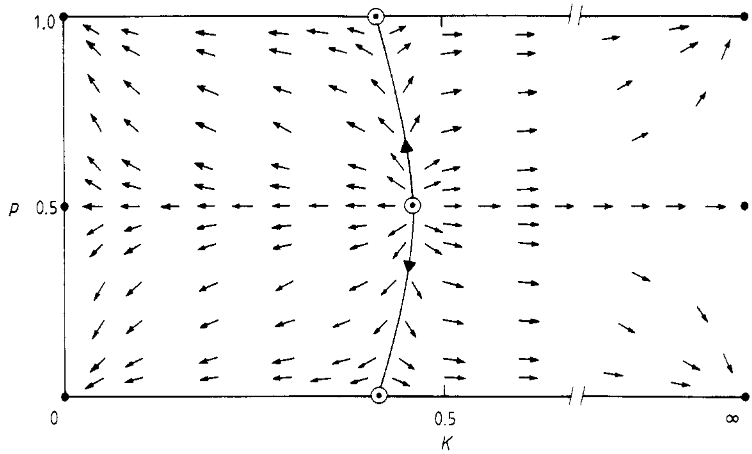

Figure 4. Flow diagram generated by the renormalisation group transformations (7) and (8). The points on the critical manifold (full curve) flow into the fixed points (⦿) on the $p=1$ and $p=0$ lines, and not to the central fixed point $p=\frac{1}{2}$. The trivial fixed points (●) are also shown.

grounds we may conclude that the Eden aggregates on the ordinary square lattice and on the SML are in different universality classes.

In this letter we have contrasted the Eden growth processes on the ordinary square lattice and on the square Manhattan lattice (SML). For this purpose we had to extend the standard Eden model in such a way that each act of growth should occur via a single lattice bond. Consequently, the extended model can be studied on both types of lattices. Applying the PSRG technique (Gould *et al* 1983) we have first compared the two versions of the model on the ordinary square lattice. In this case, our results show that the fractal dimension D_{EE} of the extended Eden model should not be smaller than the fractal dimension D_E of the standard Eden model. Thus, we can put forward the inequality $D_E \leqslant D_{EE} \leqslant 2$, which may be provocative to those who are trying to provide an exact proof for the numerical finding $D_E=2$ (Peters *et al* 1979).

Our second comparison consisted in studying universality classes of the extended Eden model on the SML and on the ordinary square lattice. We have shown that there are two different universality classes, and thereby there may be two different fractal dimensions, one (D_{EM}) for the Eden aggregates on the SML and the other (D_{EE}) for the aggregates on the ordinary square lattice. The obtained results suggest the inequality $D_{EM} < D_{EE}$ which provokes our intuition. Indeed, the SML is a totally connected lattice (any site can be reached from any other site), and one cannot easily see why the Eden aggregates on the SML should be less compact than the corresponding aggregates on the ordinary square lattice. However, it seems very likely that the initial preference of two directions of growing (instead of four), found by the seed particle placed on a site of the SML and retained partially in the further growing, acts like the tip priority factor R in the case of the electrical breakdown model introduced by Sawada *et al* (1982). These authors modified the Eden model so that, instead of weighting all the perimeter sites equally, they allowed the perimeter sites on the tip of the growing cluster to have an increased probability (parametrised by R) of being occupied. For sufficiently large R the computer simulation (Sawada *et al* 1982) confirmed that the corresponding aggregates have a fractal dimension D_S definitely less than the spatial dimension ($d=2$), and hence one may write $D_S < D_{EE}$, which is analogous to the preceding inequality, i.e. $D_{EM} < D_{EE}$.

At the end, we would like to comment on the RG scheme (Gould *et al* 1983, Green 1984) which has been adopted in this letter. We do acknowledge the observation that this scheme is not yet quantitatively reliable (Vannimenus *et al* 1984), and we do not claim that the found values of the fractal dimensions are firmly established. However, we assert that the qualitative results which concern relations between the universality classes (and relations between the fractal dimensions) are plausible, and we expect them to be vindicated by further approaches. With regard to the question of a proper generating function within the RG approach (Nakanishi and Family 1984), we have tacitly assumed the grand canonical representation of the type outlined by Prentis (1984) (with Z_N being the total number of N-site aggregates). We could not accept the generating function proposed by Nakanishi and Family (1984) as we did not find it amenable to the two-parameter PSRG technique, particularly because it elicited somewhat discouraging results in the case of the standard Eden model (Nakanishi and Family 1984). Of course, there is a danger that, within the grand caonical representation, the critical fugacity may be equal to zero (because of very large Z_N), but even if it were the case it should be possible, by dealing with finite clusters (finite N), to establish correct qualitative results. In short, because of these methodological questions we challenge a computer simulation of the Eden model on the SML. Such a study, to our knowledge, has not yet been undertaken, and it should not be more difficult than the similar study of the Eden model (Peters *et al* 1979) on the ordinary square lattice.

One of us (AC) wishes to thank the Department of Physics and Meteorology, University of Belgrade, for the stimulating atmosphere and hospitality. The other (SM) is most grateful to Professor H E Stanley for many useful references and reprints.

Note added in proof. We are grateful to Professor J Vannimenus who kindly drew our attention to the fact that the standard Eden model should be termed the 'Eden model for physicists', whereas the extended Eden model coincides with the original Eden model (Eden 1961).

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